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# NEW VERSION OF THE THERMODYNAMICALLY CONSISTENT MODEL OF MAXWELL VISCOSITY

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*Formalization of the evolutionary equations of continuum mechanics in the form of a Galilean-invariant nondivergent hyperbolic system is described. Special attention is paid to supplementing the system by additional equations required for validity of the conservation laws. A new version of Maxwell relaxation terms is proposed which is consistent with the additional equations and ensures gauge invariance.*

**Key words:** *reproducing potential, hyperbolicity, Burgers tensor, gauge invariance.*

**Introduction.** This work is a continuation of previous studies (see [1–3]) dealing with the thermodynamically consistent equations invariant under Galilean transformations.

The term thermodynamically consistent implies the existence of a divergent equality compatible with the equations, whose right side is nonnegative, and if the equations describe a dissipative process, it is strictly positive. This equality should model the law of entropy nondecrease.

It is usually assumed that in the absence of dissipative processes or if these processes are of nondiffusion relaxation nature, the equations describing the behavior of continuous media are hyperbolic. The laws of conservation of momentum and energy, as a rule, are included in the system. For these reasons, *of hyperbolic systems composed of conservation laws* are distinguished as a special object of mathematical analysis.

In [4, 5], schematization of such kind was studied for the elasticity equations with Maxwell relaxation modeling irreversible plastic deformation. When preparing a book [5] to translation into English [6], I noticed some inaccuracies [especially in the schematization from the concluding chapter, which was introduced only in the second edition in 1997) in implementing the program of setting up hyperbolic systems from conservation laws. The urgency of the work did not allowed me to clarify the reason for these inaccuracies. Therefore, reducing the last chapter, I prepared an "appendix" to the English translation, in which I attempted to outline pathways to eliminate the indicated inaccuracies.

It is customary that beginning to set up a system of equations to describe the processes of interest, one first needs to ensure hyperbolicity of the system. In other words, it is necessary to achieve that in quasilinear equations, the coefficient matrices be symmetric. (The coefficients at derivatives with respect to time $t$ should form a positive definite matrix.) For hyperbolic equations, the local Cauchy problem is correctly formulated for sufficiently smooth initial data.

Relaxation dissipative processes are described using special right sides of equations but they cannot be introduced into all equations of the system. *Some of the equations should have zero right sides* because only meeting this requirement is it possible to ensure validity of the laws of conservation of energy and momentum for the solutions. The divergent equalities describing these laws are not included in the system and *do not hold for all of its solutions* but only for those which correspond to the initial data subject *to additional conditions* in the form of some equalities.

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Implementation of the approach outlined in the indicated "appendix" requires revising our previously developed viscoelastic deformation models. In particular, greater attention should be paid to the choice of equations that include Maxwell relaxation terms. The present paper describes approaches to constructing viscoelastic models taking into account the above remarks.

In Sec. 1, we recall the necessary information from [1, 2] on Galilean-invariant hyperbolic systems and consider some relatively simple systems that will be used as component parts for further construction.

In Sec. 2, it is shown how these component parts can be combined into compound hyperbolic systems compatible with the conservation laws. The given systems satisfy the mathematical requirements formulated above.

Of course, use of the constructed equations to model the behavior of particular media requires detailed physical studies of the choice of the equations of state and dissipation laws and computational and full-scale experiments. We only propose a possible mathematical scheme for externalizing the results of studies of particular media.

It should be noted that such studies have been actively and very successfully performed by researches from Tomsk under the guidance of Academician Panin. The field theory of defects at the mesolevel developed by Panin, Grinyaev, Chertova et al., (see [7–9]) is based on subtle experiments. This theory leads to equations that can be included in the abstract scheme described herein.

The constructions below are performed invoking the gauge invariance of the equations (which is widely used recently [10–12]) relating microscopic defects to stresses. Considering geometrical and "effective elastic" strains found from the stress field, we relate the inelastic stress component not to the difference between the geometrical and effective strains but to the Burgers tensor of the latter, for which the effective strains are potentials. The possibility of ambiguous choice of these potentials does not influence the stresses, which implies *gauge invariance*. The above-mentioned drawbacks of our previous studies stemmed from the fact that gauge invariance was not ensured in them.

The concluding section (Sec. 3) deals with another two mathematically consistent versions of modeling inelastic processes. One of them is based on the "superfluidity" phenomenon used by Dorovskii (see [13, 14]), and the other was proposed and briefly described in the author's previous paper [15, 16].

**1. Examples of Hyperbolic Equations Compatible with Additional Conservation Laws.** Describing the hyperbolic systems necessary to us, we first give a list of "component parts" from which they are assembled.

It has been shown [1] that equations of the following divergent form

$$\frac{\partial L_{q_0}}{\partial t} + \frac{\partial (u_k L)_{q_0}}{\partial x_k} = 0,$$

$$\frac{\partial L_{u_i}}{\partial t} + \frac{\partial (u_k L)_{u_i}}{\partial x_k} = 0, \qquad \frac{\partial L_{p_j}}{\partial t} + \frac{\partial (u_k L)_{p_j}}{\partial x_k} = 0$$

($i, k = 1, 2, 3$ and $u_k$ are the velocity components) are Galilean-invariant if the reproducing potential

$$L = L(q_0 - u_i u_i/2, p_1, p_2, \ldots)$$

is invariant under rotations. It was assumed that under rotations, the vector made up of the components $p_j$ is transformed under any representation of the group $SO(3)$ [or $SU(2)$]. Subsequently, it turned out [3] that replacing $q_0$ by several unknowns $q_1$, $q_2, \ldots, q_m$ linked to the reproducing potential by the same relation

$$L = L(q_1 - u_i u_i/2, q_2 - u_i u_i/2, \ldots, q_m - u_i u_i/2, p_1, p_2, \ldots),$$

we can also construct a Galilean-invariant system of equations:

$$\frac{\partial L_{q_l}}{\partial t} + \frac{\partial (u_k L)_{q_l}}{\partial x_k} = 0,$$

$$\frac{\partial L_{u_i}}{\partial t} + \frac{\partial (u_k L)_{u_i}}{\partial x_k} = 0, \qquad \frac{\partial L_{p_j}}{\partial t} + \frac{\partial (u_k L)_{p_j}}{\partial x_k} = 0.$$

In [3], the role of such $q_l$ was played by the chemical potentials of the various substances or phases constituting the element of the medium. In this case, $L_{q_l}$ proved to be the partial densities of the constituents and the total density of the medium $\rho$ was expressed as the sum $\rho = \sum\limits_l L_{q_l}$.

It is easy to verify that the quasilinear form

$$L_{r_i r_j} \frac{\partial r_j}{\partial t} + M^{(k)}_{r_i r_j} \frac{\partial p_j}{\partial x_k} = 0$$

of the equations

$$\frac{\partial L_{r_i}}{\partial t} + \frac{\partial M^{(k)}_{r_i}}{\partial x_k} = 0$$

has symmetric coefficient matrices. If $L$ is a convex function of its arguments, i.e., if $L_{r_i r_j}$ form a positive definite matrix, the equations considered are symmetric hyperbolic by "Friedrichs's" definition.

More complex Galilean-invariant systems (see [2]) are constructed by supplementing the given forms of equations by additional terms that include only spatial derivatives of the unknown functions. The coefficients in these additional terms are determined from the reproducing potential $L$. The added terms should necessarily be invariant under rotations of the coordinate system. If this condition is satisfied, Galilean invariance will be ensured.

Below we give some examples of systems of equations constructed by the method described above and formulate some constraints on them. These constraints result from ensuring compatibility of the equations with the additional relations required for the validity of the laws of conservation of momentum and energy.

The first example, which can be used as the canonical form of the equations of magnetohydrodynamics, has the following form:

$$\frac{\partial L_{q_0}}{\partial t} + \frac{\partial (u_k L)_{q_0}}{\partial x_k} = 0; \tag{1.1a}$$

$$\frac{\partial L_{r_i}}{\partial t} + \frac{\partial (u_k L)_{r_i}}{\partial x_k} - L_{r_k} \frac{\partial u_i}{\partial x_k} = 0; \tag{1.1b}$$

$$\frac{\partial L_{u_i}}{\partial t} + \frac{\partial (u_k L)_{u_i}}{\partial x_k} - L_{r_k} \frac{\partial r_i}{\partial x_k} = 0 \tag{1.1c}$$

[$L = L(q_0 + u_i u_i/2, r_i r_i)$, where $i, k = 1, 2, 3$]. Here nondivergent terms are added to Eqs. (1.1b) and (1.1c). The symmetry of the coefficient matrix is obviously not broken by the addition of these terms: to (1.1b) we added derivatives of $u_i$, and to (1.1c), derivatives of $r_i$ with identical coefficients $L_{r_k}$. We note that Eqs. (1.1c) do not have the form of conservation laws and, hence, they cannot be treated as the law of conservation of momentum. This defect is corrected as follows. Equalities (1.1a) and (1.1b) imply that

$$\frac{\partial L_{r_i}}{\partial t} + \frac{\partial \left(u_k L_{r_i} - u_i L_{r_k}\right)}{\partial x_k} + u_i \frac{\partial L_{r_k}}{\partial x_k} = 0,$$

$$\frac{\partial}{\partial t}\Big(\frac{\partial L_{r_i}}{\partial x_i}\Big) + \frac{\partial}{\partial x_k}\Big(u_k \frac{\partial L_{r_i}}{\partial x_i}\Big) = 0, \qquad \frac{\partial}{\partial t}\Big(\frac{1}{L_{q_0}} \frac{\partial L_{r_i}}{\partial x_i}\Big) + u_k \frac{\partial}{\partial x_k}\Big(\frac{1}{L_{q_0}} \frac{\partial L_{r_i}}{\partial x_i}\Big) = 0.$$

From this it follows that system (1.1) is compatible with the additional equality

$$\frac{\partial L_{r_i}}{\partial x_i} = 0. \tag{1.2}$$

For solutions (1.1) subject to condition (1.2), Eq. (1.1b) can be replaced by the conservation law

$$\frac{\partial L_{u_i}}{\partial t} + \frac{\partial \left[(u_k L)_{u_i} - r_i L_{r_k}\right]}{\partial x_k} = 0. \tag{1.3}$$

In magnetohydrodynamics, it is the law of conservation of momentum. Multiplying Eqs. (1.1b) and (1.1c) by $q_0$ and $r_i$, respectively, and Eq. (1.3) by $u_i$ and summing the products, we arrive at one more conservation law (conservation of energy):

$$\frac{\partial \mathcal{E}}{\partial t} + \frac{\partial \left[u_k(\mathcal{E} + L) - u_i r_i L_{r_k}\right]}{\partial x_k} = 0, \qquad \mathcal{E} = q_0 L_{q_0} + r_i L_{r_i} + u_i L_{u_i} - L. \tag{1.4}$$

The above example shows that for the symmetric hyperbolic system (1.1), the conservation laws (1.3) and (1.4) should not necessarily be satisfied for all of its solutions. They are valid subject to the additional condition (1.2). For the compatibility of (1.2) with the basic equations, it is essential that (1.1b) and (1.1c) have zero right sides. Any right sides that are smooth functions of the unknowns $q_0$, $r_i$, and $u_i$ do not disturb the hyperbolicity of the system but they do not always enable one to establish the compatibility of (1.1) with (1.2), i.e., to justify the conservation laws (1.3) and (1.4).

The above example is easy to extend to model elastic processes in isotropic media. For this, one only needs to replace the vector variable with the components $r_i$ by a tensor variable $r_{ij}$ (the tensor $r_{ij}$ should not necessarily be symmetric).

Let us give the corresponding hyperbolic system:

$$\frac{\partial L_{q_0}}{\partial t} + \frac{\partial (u_k L)_{q_0}}{\partial x_k} = 0,$$

$$\frac{\partial L_{r_{ij}}}{\partial t} + \frac{\partial (u_k L)_{r_{ij}}}{\partial x_k} - L_{r_{kj}} \frac{\partial u_i}{\partial x_k} = 0, \qquad \frac{\partial L_{u_i}}{\partial t} + \frac{\partial (u_k L)_{u_i}}{\partial x_k} - L_{r_{kj}} \frac{\partial r_{ij}}{\partial x_k} = 0, \tag{1.5}$$

compatible with the additional equality

$$\frac{\partial L_{r_{ij}}}{\partial x_i} = 0, \tag{1.6}$$

which ensures the validity of the conservation laws

$$\frac{\partial L_{u_i}}{\partial t} + \frac{\partial [(u_k L)_{u_i} - r_{ij} L_{r_{kj}}]}{\partial x_k} = 0, \qquad \frac{\partial \mathcal{E}}{\partial t} + \frac{\partial [u_k(\mathcal{E} + L) - u_i r_{ij} L_{r_{kj}}]}{\partial x_k} = 0,$$

$$\mathcal{E} = q_0 L_{q_0} + r_{ij} L_{r_{ij}} + u_i L_{u_i} - L. \tag{1.7}$$

Interpretation of Eqs. (1.7) and some others presented below will be dealt with in Sec. 2.

The construction described above can be generalized by introducing one more new vector equation (conservation law)

$$\frac{\partial L_{r_i}}{\partial t} + \frac{\partial (u_k L)_{r_i}}{\partial x_k} = 0, \tag{1.8}$$

and supplementing the last equation in (1.5) by the nonzero right side

$$\frac{\partial L_{u_i}}{\partial t} + \frac{\partial (u_k L)_{u_i}}{\partial x_k} - L_{r_{kj}} \frac{\partial r_{ij}}{\partial x_k} = r_{ij} L_{r_j}. \tag{1.9}$$

The first and second equalities in (1.5) imply that

$$\frac{\partial}{\partial t}\Big(\frac{1}{L_{q_0}} \frac{\partial L_{r_{ij}}}{\partial x_i}\Big) + u_k \frac{\partial}{\partial x_k}\Big(\frac{1}{L_{q_0}} \frac{\partial L_{r_{ij}}}{\partial x_i}\Big) = 0,$$

and Eq. (1.8), together with the equation in the first line of (1.5) imply that

$$\frac{\partial}{\partial t}\Big(\frac{1}{L_{q_0}} L_{r_j}\Big) + u_k \frac{\partial}{\partial x_k}\Big(\frac{1}{L_{q_0}} L_{r_j}\Big) = 0.$$

Therefore, system (1.5) supplemented by (1.8) is compatible with the relation

$$\frac{\partial L_{r_{ij}}}{\partial x_i} - L_{r_j} = 0, \tag{1.10}$$

which, in this example, substitutes for (1.6). If relation (1.10) is satisfied, Eq. (1.9) ensures the validity of the conservation laws:

$$\frac{\partial L_{u_i}}{\partial t} + \frac{\partial [(u_k L)_{u_i} - r_{ij} L_{r_{kj}}]}{\partial x_k} = 0, \qquad \frac{\partial \mathcal{E}}{\partial t} + \frac{\partial [u_k(\mathcal{E} + L) - u_i r_{ij} L_{r_{kj}}]}{\partial x_k} = 0,$$

$$\mathcal{E} = q_0 L_{q_0} + r_{ij} L_{r_{ij}} + r_i L_{r_j} - L. \tag{1.11}$$

Let us write another example similar to (1.5):

$$\frac{\partial L_n}{\partial t} + \frac{\partial\,(u_k L_n + \varepsilon_{ijk} b_{ij})}{\partial x_k} = 0,$$

$$\frac{\partial L_{b_{ij}}}{\partial t} + \frac{\partial\,(u_k L_{b_{ij}} + \varepsilon_{ijk} n)}{\partial x_k} - L_{b_{kj}}\,\frac{\partial u_i}{\partial x_k} = 0, \qquad \frac{\partial L_{u_i}}{\partial t} + \frac{\partial\,(u_k L)_{u_i}}{\partial x_k} - L_{b_{kj}}\,\frac{\partial b_{ij}}{\partial x_k} = 0 \tag{1.12}$$

($\varepsilon_{ijk}$ is the Levi-Civita symbol; it is equal to zero for equal values of $i$, $j$, and $k$ and to $\pm 1$ for different $i$, $j$, and $k$, according to the evenness or oddness of index substitution).

As above, it is verified that system (1.12) is compatible with the additional equality

$$\frac{\partial L_{b_{kj}}}{\partial x_k} = 0, \tag{1.13}$$

which allows Eq. (1.12) to be written in divergent form

$$\frac{\partial L_{u_i}}{\partial t} + \frac{\partial\,[(u_k L)_{u_i} - b_{ij} L_{b_{kj}}]}{\partial x_k} = 0, \tag{1.14}$$

and to derive one more conservation law

$$\frac{\partial \mathcal{E}}{\partial t} + \frac{\partial\,[u_k(\mathcal{E} + L) - u_i b_{ij} L_{b_{kj}} + n\varepsilon_{ijk} b_{ij}]}{\partial x_k} = 0, \tag{1.15}$$

$$\mathcal{E} = nL_n + b_{ij} L_{b_{ij}} + u_i L_{u_i} - L.$$

In the conclusion of Sec. 1, we give a system set up of the previous equations as component parts. In addition, we include a few more divergent equations and new variables — the temperature $T$ and the four-index tensor $p_{ijlm}$, which is intended for the elastic moduli if a nonisotropic medium is considered:

$$\frac{\partial L_{q_0}}{\partial t} + \frac{\partial\,(u_k L)_{q_0}}{\partial x_k} = 0, \qquad \frac{\partial L_{r_{ij}}}{\partial t} + \frac{\partial\,(u_k L)_{r_{ij}}}{\partial x_k} - L_{r_{kj}}\,\frac{\partial u_i}{\partial x_k} = 0,$$

$$\frac{\partial L_{r_i}}{\partial t} + \frac{\partial\,(u_k L)_{r_i}}{\partial x_k} = 0, \qquad \frac{\partial L_n}{\partial t} + \frac{\partial\,(u_k L_n + \varepsilon_{ijk} b_{ij})}{\partial x_k} = 0,$$

$$\frac{\partial L_{b_{ij}}}{\partial t} + \frac{\partial\,(u_k L_{b_{ij}} + \varepsilon_{ijk} n)}{\partial x_k} - L_{b_{kj}}\,\frac{\partial u_i}{\partial x_k} = 0, \qquad \frac{\partial L_{p_{ijlm}}}{\partial t} + \frac{\partial\,(u_k L)_{p_{ijlm}}}{\partial x_k} = 0, \tag{1.16}$$

$$\frac{\partial L_{u_i}}{\partial t} + \frac{\partial\,(u_k L)_{u_i}}{\partial x_k} - L_{r_{kj}}\,\frac{\partial r_{ij}}{\partial x_k} - L_{b_{kj}}\,\frac{\partial b_{ij}}{\partial x_k} = r_{ij} L_{r_j}, \qquad \frac{\partial L_T}{\partial t} + \frac{\partial\,(u_k L)_T}{\partial x_k} = 0.$$

This system is compatible with the additional equalities

$$\frac{\partial L_{r_{ij}}}{\partial x_i} - L_{r_j} = 0, \qquad \frac{\partial L_{b_{ij}}}{\partial x_i} = 0, \tag{1.17}$$

whose satisfaction ensures that for the solutions, the laws of conservation of momentum and energy are valid:

$$\frac{\partial L_{u_i}}{\partial t} + \frac{\partial\,[(u_k L)_{u_i} - r_{ij} L_{r_{kj}} - b_{ij} L_{b_{kj}}]}{\partial x_k} = 0,$$

$$\frac{\partial \mathcal{E}}{\partial t} + \frac{\partial\,[(u_k(\mathcal{E} + L) + n b_{ij}\varepsilon_{ijk} - u_i(r_{ij} L_{r_{kj}} + b_{ij} L_{b_{kj}})]}{\partial x_k} = 0. \tag{1.18}$$

As in the previous examples, $\mathcal{E}$ denotes the Legendre transform of the reproducing potential $L$ over all of its arguments:

$$\mathcal{E} = q_0 L_{q_0} + r_i L_{r_i} + r_{ij} L_{r_{ij}} + nL_n + b_{ij} L_{b_{ij}} + p_{ijkl} L_{p_{ijkl}} + TL_T + u_i L_{u_i} - L. \tag{1.19}$$

**2. Interpretation of Equations and Validity of Including Dissipative Terms in Them.** Equations (1.16)–(1.18) given at the end of Sec. 1 can be used to describe elastic processes in a moving continuum. In this case, it should be assumed that $L_{u_i} = \rho u_i$ and that the equations

$$\frac{\partial L_{q_0}}{\partial t} + \frac{\partial\,(u_k L)_{q_0}}{\partial x_k} = 0, \qquad \frac{\partial L_{r_{ij}}}{\partial t} + \frac{\partial\,(u_k L)_{r_{ij}}}{\partial x_k} - \frac{\partial u_i}{\partial x_k}\,L_{r_{kj}} = 0$$

are representations of the continuity equation and the evolution equation for the distortion $c_{ij} = \partial x_i/\partial x_{j0}$, respectively, in a different notation ($L_{q_0} = \rho$ and $L_{r_{ij}} = \rho c_{ij}$), ($x_i$ are Euler coordinates and $x_{j0}$ are Lagrangian coordinates):

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u_k)}{\partial x_k} = 0, \qquad \frac{\partial (\rho c_{ij})}{\partial t} + \frac{\partial (u_k \rho c_{ij})}{\partial x_k} - \frac{\partial u_i}{\partial x_k}\, \rho c_{kj} = 0$$

(for this issue see [5, 6, 17]). The vector $L_{r_j} = \partial \rho c_{ij}/\partial x_i$ may be nonzero if in the initial data $\rho \neq$ const and $c_{ij} \neq \delta_{ij}$. If $\rho c_{ij} = r_{ij} =$ const at $t = 0$, the equality $\partial \rho c_{ij}/\partial x_i = r_i = 0$ also holds for $t > 0$, as follows from the above reasoning. The introduced variable $T$ is the temperature; $L_T = \rho S$ is the entropy per unit volume, for which the conservation law is described by the last equation in (1.16). The tensor $p_{ijkl}$ can be treated as the elastic modulus tensor.

The variables $b_{ij}$ are introduced into the system to describe the continual dislocation field. In this case, the tensor composed of the values of $L_{b_{ij}}$ plays the role of a transposed Burgers tensor. The dislocation field characterizes the difference between the physical Lagrangian distortion $A^{\text{eff}}$ and the geometrical distortion $A = C^{-1}$ ($C$ is a matrix with elements $c_{ij} = \partial x_i/\partial x_{j0}$ and $a_{ij} = \partial x_{i0}/\partial x_j$). For the geometrical Lagrangian distortion, the Burgers tensor $\mathcal{B}_{ji} = \varepsilon_{kli}\, \partial a_{jl}/\partial x_k$ is obviously equal to zero. The tensor $L_{b_{ij}} = \mathcal{B}^{\text{eff}}_{ji} = \varepsilon_{kli}\, \partial a^{\text{eff}}_{jl}/\partial x_k$ is no longer necessarily equal to zero but it should necessarily satisfy the additional equation $\partial L_{b_{ij}}/\partial x_i = 0$ whatever the law of variation of $A^{\text{eff}}$ due to inelastic processes, i.e., the processes in which the stress varies not only as a result of the geometrical deformation described by variation in the density $\rho$ and the tensor $L_{r_{ij}} = \rho c_{ij}$. For this reason, the proposed equations for $L_{b_{ij}}$ contain derivatives of the corresponding "chemical potential" $n$ governing the evolution of the Burgers tensor. These derivative are introduced into the equations so as to ensure validity of the equality $\partial L_{b_{ij}}/\partial x_i = 0$. Here we do not give equations for $A^{\text{eff}}_{ij}$, thus ensuring "gauge invariance," i.e., independence of the deformation process and the magnitude of the stress field $\sigma_{ik} = L - r_{ij} L_{r_{kj}} - b_{ij} L_{b_{kj}}$ on the particular choice of $A^{\text{eff}}_{ij}$ compatible with the values of $L_{b_{ij}}$ participating in the equations ($A^{\text{eff}}_{ij}$ are potentials with respect to the Burgers tensor).

Terms accounting for dissipative processes were not included in system (1.16) and, hence, it does not model them. We now include relaxation right sides in some of the equations. This should be done with caution lest these right sides break the compatibility of the equations with the additional equalities $\partial L_{r_{ij}}/\partial x_i = r_j$ and $\partial L_{b_{ij}}/\partial x_i = 0$ required to ensure conservation of momentum and energy. We modify only three lines of system (1.16), which now become

$$\frac{\partial L_n}{\partial t} + \frac{\partial (u_k L_n + \varepsilon_{ijk} b_{ij})}{\partial x_k} = -\frac{\Phi_n}{\tau_1},$$

$$\frac{\partial L_{p_{ijlm}}}{\partial t} + \frac{\partial (u_k L)_{p_{ijlm}}}{\partial x_k} = -\frac{\Psi_{p_{ijlm}}}{\tau_2}, \qquad \frac{\partial L_T}{\partial t} + \frac{\partial (u_k L)_T}{\partial x_k} = \frac{1}{T}\Big(\frac{n\Phi_n}{\tau_1} + \frac{p_{ijlm}\Psi_{p_{ijlm}}}{\tau_2}\Big). \tag{2.1}$$

In (2.1), $\Phi$ and $\Psi$ are dissipative functions for which $n\Phi_n \geqslant 0$ and $p_{ijlm}\Psi_{p_{ijlm}} \geqslant 0$, and the positive parameters $\tau_1 > 0$ and $\tau_2 > 0$, which depend on the state of the medium, characterize the dissipation rate. The right side of the latter from the modified lines is chosen such that the addition of dissipative terms does not break the energy conservation law and that the entropy increases.

Here we allowed the parameters $p_{ijlm}$, which describe the elastic properties of the medium, to relax and vary, whereas the geometrical parameters $\rho = L_{q_0}$ and $\rho c_{ij} = L_{r_{ij}}$ cannot relax in the proposed model. The model described in papers [4, 5] and in the main part of their English version [6] was based on incorporation of Maxwell relaxation in the equations for $\rho c_{ij} = L_{r_{ij}}$, which were related to "effective elastic" rather than real geometrical strains. As was noted in the introduction, this circumstance led to violation of the necessary conservation laws. The constructions described in the present paper lead to a new version of the Maxwell viscoelastic model, in which the indicated drawback is eliminated.

We note that here we do not list all permissible versions of accounting for the dissipative processes compatible with gauge invariance. If, for example, the fifth equality of (1.16) is not written with zero right side but is represented as ($\text{æ} \geqslant 0$)

$$\frac{\partial L_{b_{ij}}}{\partial t} + \frac{\partial (u_k L_{b_{ij}} + \varepsilon_{ijk} n)}{\partial x_k} - L_{b_{kj}} \frac{\partial u_i}{\partial x_k} = \varepsilon_{i\alpha\beta} \frac{\partial}{\partial x_\alpha}\Big(\text{æ}\, \varepsilon_{\beta\gamma\delta} \frac{\partial b_{\delta j}}{\partial x_\gamma}\Big), \tag{2.2}$$

then it is easy to verify that it is compatible with the relation $\partial L_{b_{ij}}/\partial x_i = 0$ required to us. The right side of (2.2) is introduced to model the diffusion of dislocation defects. Along with the indicated modification of Eqs. (1.16), the zero right sides of the energy equation [the second in (1.18)] and the entropy equation should be replaced, respectively, by

$$-\frac{\partial}{\partial x_\alpha}\Big[b_{ij}æ\Big(\varepsilon_{\alpha i\beta}\varepsilon_{\beta\alpha\delta}\,\frac{\partial b_{\delta j}}{\partial x_\gamma}\Big)\Big], \qquad \frac{æ}{T}\sum_{\beta,j}\Big(\varepsilon_{\beta\gamma\delta}\,\frac{\partial b_{\delta j}}{\partial x_\gamma}\Big)^2.$$

These right sides describe the additional energy flux and the entropy increment due to diffusion of defects.

**3. A Few Other Generalizations.** We give one more version of thermodynamically consistent hyperbolic equations for a two-phase medium, one of whose phases behaves like a superfluid liquid. This model was inspired by Dorovskii's papers [13, 14], although it does not follows them literally. The model uses two chemical potentials $q_0$ and $q_1$, which refer to the elastoplastic and superfluid phases, respectively. In this case, the partial densities of these phases are described using the derivatives $L_{q_0}$ and $L_{q_1}$ of the reproducing potential $L$. The partial densities satisfy the equations

$$\frac{\partial L_{q_0}}{\partial t} + \frac{\partial\,(u_k L)_{q_0}}{\partial x_k} = -(L_{q_0} + L_{q_1})\sum_s \nu_0^{(s)}\Big(\frac{\nu_0^{(s)}q_0 + \nu_1^{(s)}q_1}{\tau^{(s)}}\Big),$$

$$\frac{\partial L_{q_1}}{\partial t} + \frac{\partial\,(u_k L_{q_1} + v_k)}{\partial x_k} = -(L_{q_0} + L_{q_1})\sum_s \nu_1^{(s)}\Big(\frac{\nu_0^{(s)}q_0 + \nu_1^{(s)}q_1}{\tau^{(s)}}\Big), \tag{3.1}$$

whose right sides model the exchange reactions between the phases. There may be several such reactions, and each of them is characterized by particular stoichiometric coefficients $\nu_0^{(s)}$ and $\nu_1^{(s)}$ ($\nu_0^{(s)} + \nu_1^{(s)} = 0$) and particular parameters $\tau^{(s)}$ determining the reaction rate. In this formulation, we follow the schematizations described in [3]. Equations (3.1) imply the validity of the continuity equation

$$\frac{\partial\,(L_{q_0} + L_{q_1})}{\partial t} + \frac{\partial\,[u_k(L_{q_0} + L_{q_1}) + v_k]}{\partial x_k} = 0. \tag{3.2}$$

The components $v_k$ participating in (3.1) and (3.2) specify the mass velocity of the superfluid phase relative to the motion of the medium.

As in the previous Secs. 1 and 2, the elastic or elastoplastic phase is described by the equations

$$\frac{\partial L_{r_{ij}}}{\partial t} + \frac{\partial\,(u_k L)_{r_{ij}}}{\partial x_k} - \frac{\partial u_i}{\partial x_k}\,L_{r_{kj}} = 0,$$

$$\frac{\partial L_{r_i}}{\partial t} + \frac{\partial\,(u_k L)_{r_i}}{\partial x_k} = 0, \qquad \frac{\partial L_n}{\partial t} + \frac{\partial\,(u_k L_n + \varepsilon_{ijk}b_{ij})}{\partial x_k} - \frac{\Phi_n}{\tau_1} = 0, \tag{3.3}$$

$$\frac{\partial L_{b_{ij}}}{\partial t} + \frac{\partial\,(u_k L_{b_{ij}} + \varepsilon_{ijk}n)}{\partial x_k} - L_{b_{kj}}\,\frac{\partial u_i}{\partial x_k} = 0, \qquad \frac{\partial L_{p_{ijlm}}}{\partial t} + \frac{\partial\,(u_k L)_{p_{ijlm}}}{\partial x_k} = -\frac{\Psi_{p_{ijlm}}}{\tau_2}.$$

To close this system, it should be supplemented by the equations governing the mass flux with the components $v_i$, the equations for the entropy $L_T$, and the equations for the momentum components $L_{u_i}$. The latter will be considered later, and the remaining are chosen as follows:

$$\frac{\partial L_{v_i}}{\partial t} + \frac{\partial\,(u_k L)_{v_i}}{\partial x_k} - \Big(L_{v_i}\,\frac{\partial u_k}{\partial x_k} - L_{v_k}\,\frac{\partial u_k}{\partial x_i}\Big) + \frac{\partial q_1}{\partial x_k} = 0,$$

$$\frac{\partial L_T}{\partial t} + \frac{\partial\,(u_k L)_T}{\partial x_k} = \frac{1}{T}\Big[(L_{q_0} + L_{q_1})\sum_s \tau^{(s)}(\nu_0^{(s)}q_0 + \nu_1^{(s)}q_1)^2 + \frac{n\Phi_n}{\tau_1} + \frac{p_{ijlm}\Psi_{p_{ijlm}}}{\tau_2}\Big] \geqslant 0. \tag{3.4}$$

As is shown, for example, in [3], the first line in (3.4) is compatible with the additional equations

$$\frac{\partial L_{v_i}}{\partial x_k} - \frac{\partial L_{v_k}}{\partial x_i} = 0. \tag{3.5}$$

Equalities (3.3) are compatible with the equations

$$\frac{\partial L_{r_{ij}}}{\partial x_i} = L_{r_i}, \qquad \frac{\partial L_{b_{ij}}}{\partial x_i} = 0. \tag{3.6}$$

The nondivergent momentum equation is written as

$$\frac{\partial L_{u_i}}{\partial t} + \frac{\partial (u_k L)_{u_i}}{\partial x_k} - L_{r_{kj}} \frac{\partial r_{ij}}{\partial x_k} - L_{b_{kj}} \frac{\partial b_{ij}}{\partial x_k} + L_{v_i} \frac{\partial v_k}{\partial x_k} - L_{v_k} \frac{\partial v_k}{\partial x_i} = r_{ij} L_{r_j}. \tag{3.7}$$

The system set up of Eqs. (3.1), (3.3), (3.4), and (3.7) is symmetric hyperbolic if the reproducing potential is a convex function of its arguments. We shall not verify this statement. It is verified by using the same line of reasoning as above and in [2] for the component equations of our system.

By means of the additional relations (3.5) and (3.6), Eqs. (3.7) are reduced to the divergent form

$$\frac{\partial L_{u_i}}{\partial t} + \frac{\partial \left[ (u_k L)_{u_i} - r_{ij} L_{r_{kj}} - b_{ij} L_{b_{kj}} + v_k L_{v_i} - \delta_{ik} v_r L_{v_r} \right]}{\partial x_k} = 0, \tag{3.8}$$

$$\frac{\partial L_{u_i}}{\partial t} + \frac{\partial \left[ (u_k L)_{u_i} - r_{ij} L_{r_{kj}} b_{ij} L_{b_{kj}} + v_k L_{v_i} - \delta_{ik} v_r L_{v_r} \right]}{\partial x_k} = 0, \qquad (3.8)$$

which is used, as in the examples from Sec. 2 and the present chapter, to derive the divergent equality of the energy conservation law. We do not give it here. The constructed equations are Galilean-invariant if $L$ is invariant under rotations and if its dependence on $q_0$, $q_1$, and $u_i$ is given by

$$L = L(q_0 - u_i u_i/2, q_1 - u_i u_i/2, \dots).$$

We note that it is reasonable to write Eq. (3.8) with the use of the stress tensor

$$\frac{\partial L_{u_1}}{\partial t} + \frac{\partial \left[ u_k L_{u_i} - \sigma_{ik} \right]}{\partial x_k} = 0,$$

$$\sigma_{ik} = -L + r_{ij} L_{r_{kj}} + b_{ij} L_{b_{kj}} + v_k L_{v_i} - \delta_{ik} v_r L_{v_r},$$

in which the terms are the products $r_{ij} L_{r_{kj}}$, $b_{ij} L_{b_{kj}}$, $v_k L_{v_i}$, and $-\delta_{ik} v_k L_{v_k}$, each of which corresponds to a particular "component part" of the structure. In constructing the above models, we alternatively increased the number of equations and the processes described by them, each time including additional terms in the law of conservation of momentum, i.e., modifying the stress tensor $\sigma_{ik}$.

There is another method of including new unknowns in constructing thermodynamically consistent hyperbolic systems (see [15–17]). As such new unknowns, it is possible to choose the additional terms $\gamma_{ik}$ included in the stress tensor calculated by the formula

$$\sigma_{ik} = -L + r_{ij} L_{r_{kj}} + b_{ij} L_{b_{kj}} + v_k L_{v_i} - \delta_{ik} v_r L_{v_r} - \gamma_{ik}.$$

In this case, of course, it should be assumed that $\gamma_{ik}$ are included as additional arguments in the reproducing potential $L$.

Equations (3.7) now become

$$\frac{\partial L_{u_i}}{\partial t} + \frac{\partial \left[ (u_k L)_{u_i} - \gamma_{ik} \right]}{\partial x_k} - L_{r_{kj}} \frac{\partial r_{ij}}{\partial x_k} - L_{b_{kj}} \frac{\partial b_{ij}}{\partial x_k} + L_{v_i} \frac{\partial v_k}{\partial x_k} - L_{v_k} \frac{\partial v_k}{\partial x_i} = r_{ij} L_{r_j}.$$

In addition, the system should be supplemented by new equations. These equations can contain dissipative right sides:

$$\frac{\partial L_{\gamma_{ij}}}{\partial t} + \frac{\partial (u_k L)_{\gamma_{ij}}}{\partial x_k} - \frac{\partial u_i}{\partial x_i} = \frac{\Omega_{\gamma_{ij}}}{\tau_3}$$

($\Omega$ is a dissipation function), which require inclusion of the additional term $T^{-1} \gamma_{ij} \Omega_{\gamma_{ij}}$ in the right side of the entropy equation. The reader can easily verify that after the extension described above, our equations ensure that the divergent energy equation with zero right side holds. Of course, in this case it is necessary to assume that the additional conditions (3.5) and (3.6) are also valid.

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